ALGEBRA

List 1.

Binomial formula. Induction. Complex numbers

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1. Using the Newton Binomial formula, transform

$$(a)\,(2x-y)^4;\quad (b)\,\left(x+\frac{1}{x^3}\right)^3;\quad (c)\,(\sqrt{u}-\sqrt[4]{v})^8;\quad (d)\,\left(x+\frac{1}{x^3}\right)^6.$$

2. Find the coefficient at term t in the expansion

$$(a)\,(2p-3q)^7,\ t=p^2q^5;\quad (b)\,\left(\sqrt[4]{b^5}-\frac{3}{b^3}\right)^7,\ t=\sqrt[4]{b}\quad (c)\,\left(2x-\frac{1}{x}\right)^6\left(x+\frac{1}{2x}\right)^6,\ t=x^0.$$

3. Using the mathematical induction, prove the equalities:

(a) $1+2+3+\cdots+n=\frac{n(n+1)}{2}, n\in\mathbf{N}$;

(b) $1+3+5+\cdots+(2n-1)=n^2, n\in\mathbf{N}$;

(c) $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}, n\in\mathbf{N}$.

4. Using the mathematical induction, prove the inequalities:

(a) $n^3<3^n, n\in\mathbf{N}$;

(b) $\frac{1}{\sqrt{1}}+\ldots\frac{1}{\sqrt{n}}\geqslant\sqrt{n}, n\in\mathbf{N}$;

(c) $(1+a)^n\geqslant 1+na$, for any $a\geqslant -1$ and $n\in\mathbf{N}$.

5. Perform the algebraic operations and write the result in the Carthesian form $x+iy$:

$$(a)\,(i+3)-(2-3i);\quad (b)\,(1-i)(2+5i);\quad (c)\,\frac{1-3i}{2+3i};\quad (d)\,(1-i)^4.$$

6. Comparing the real and imaginary parts of both sides of the equations, solve them for real x,y:

$$(a)\,(1-i)x+(2-i)y=1+i;\quad (b)\,\frac{x}{1-i}+\frac{y}{1+i}=1+i;\quad (c)\,2x^2+iy^2=3;\quad (d)\,3x^2-2iy^2=(1+i)(i-2).$$

7. Writing z in the algebraic form $z=x+iy$, solve the equations

$$(a)\,z^2=-i;\quad (b)\,(3-2i)z=(2+i);\quad (c)\,\frac{z+1}{2+i}=\frac{3-z}{3-2i};\quad (d)\,z^2-4z+5;$$

$$(e)\,z(1+i)+\overline{z}(2-i)=1+i;\quad (f)\,i\operatorname{Re}z+\operatorname{Im}z=1+2i;\quad (g)\,z\overline{z}=(\overline{z})^2.$$

8. Find all complex numbers z which satisfy the following conditions:

$$(a)\operatorname{Re}z+\operatorname{Im}z=3;\quad (b)\operatorname{Re}(-iz)\leqslant 1;\quad (c)\operatorname{Im}((1+i)z)\leqslant 2.$$

Indicate the solution on the complex plane.

9. Using the Carthesian form of complex numbers, compute the following roots:

$$(a)\,\sqrt{1-2i};\quad (b)\,\sqrt{5-i}.$$

10. Solve the equations for complex z:

$$(a)\, z^2-z+1=0; \quad (b)\, z^2+16=0; \quad (c)\, z^4-3z^2+2=0; \quad (d)\, z^2+(1-i)z+2i=0; \quad (e)\, z^4=-1;$$

$$(f)\, z^2+4iz+1=0; \quad (g)\, z^3=(1+i)^3; \quad (h)\, (z-i)^4=(2z+1)^4.$$

11. Write the following numbers in the trigonometric form:

$$(a)\, 2i; \quad (b)\, -1+\sqrt{3}i; \quad (c)\, -2\sqrt{3}-2i; \quad (d)\, \left(\frac{1-\sqrt{3}i}{2+2\sqrt{3}i}\right)^5.$$

12. Using de Moivre's formula, compute the following powers:

$$(a)\, (1-i)^{13}; \quad (b)\, (-1+\sqrt{3}i)^{15}; \quad \left(\frac{1+i}{-1+i\sqrt{3}}\right)^{17}.$$

Give the answers in the Carthesian form.

13. Using the trigonometric form of complex numbers, compute the following roots:

$$(a)\, \sqrt[6]{-1}; \quad (b)\, \sqrt[3]{-\sqrt{3}+i}; \quad (c)\, \sqrt[6]{-64}.$$

Give the answers in the Carthesian form.

14. Solve the following equations:

$$(a)\, (z+1)^3=(z-2)^3; \quad (b)\, (z+i)^4=(1-z)^4; \quad (c)\, (2z-1)^3=(z+i)^3.$$

Give the answers in the Carthesian form.

15. Draw on the complex plane the sets of complex numbers satisfying the following conditions:

$$(a)\, |2z+i|=6; \quad (b)\, |3z-1|<3; \quad (c)\, 2\leqslant|2z+i|\leqslant 4; \quad (d)\, |z-2i|=|z+i|;$$

$$(e)\, \mathrm{Im}\,(z^3)<0; \quad (f)\, \mathrm{Re}\,(z^4)\geqslant 0; \quad (h)\, |z+1|\leqslant|\overline{z}+i|.$$